On the quantum (in)stability in cavity QED

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The stability and instability of quantum motion is studied in the context of cavity quantum electrodynamics (QED). It is shown that the Jaynes-Cummings dynamics can be unstable in the regime of chaotic walking of an atom in the quantized field of a standing wave in the absence of any other interaction with environment. This quantum instability manifests itself in strong variations of quantum purity and entropy and in exponential sensitivity of fidelity of quantum states to small variations in the atom-field detuning. It is quantified in terms of the respective classical maximal Lyapunov exponent that can be estimated in appropriate in-out experiments.

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The problem of stability of quantum dynamics has attracted a great interest by its own right and in relation to the field of quantum information and computation. Classical instability is usually defined as an exponential separation of two nearly trajectories in time with an asymptotic rate given by the maximal Lyapunov exponent λ. Perfectly isolated quantum systems are unitary and cannot be unstable in this sense even if their classical limits are chaotic [1]. It is well known that quantum coherence is destroyed due to interaction with an environment [2, 3] which is usually modeled by a heat bath with infinitely many degrees of freedom. Environment-induced decoherence causes quantum-entropy increase which is dominated by the classical Lyapunov exponents [4]. In an alternative approach [5, 6] the quantum instability is proposed to be measured by the decay of the fidelity or the overlap

$$f(\tau) = \left|\langle \Psi_1(\tau)\,|\Psi_2(\tau)\rangle\right|^2 \tag{1}$$

of two wave functions Ψ_1 and Ψ_2, identical at $\tau = 0$, that evolve under slightly different Hamiltonians.

In a number of numerical studies [7, 8, 9, 10] for a variety of classically chaotic models it has been established that the overlap decay may be algebraic, Gaussian, and exponential. The strength of perturbations in Hamiltonians and other factors determine which of these regimes prevails.

In this letter we show that instability of quantum dynamics and its exponential sensitivity to initial conditions and small variations in parameters may occur in a paradigmic cavity-QED system with a single environmental degree of freedom. To specify the problem we consider the standard model in cavity QED Jaynes-Cummings Hamiltonian [11]

$$\hat{H} = \frac{\hat{p}^2}{2m_a} + \frac{1}{2}\hbar\omega_a\hat{\sigma}_z + \hbar\omega_f\hat{a}^\dagger\hat{a} - \\ - \hbar\Omega_0\left(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right)\cos k_f\hat{x}, \tag{2}$$

which describes the interaction between a two-level atom with lower, $|1\rangle$, and upper, $|2\rangle$, states, the transition frequency ω_a, and the Pauli operators $\hat{\sigma}_{\pm,z}$ and a quantized electromagnetic-field mode with creation, $\hat{a}^\dagger$, and annihilation, $\hat{a}$ operators forming a standing wave with the frequency ω_f and the wave vector k_f in an ideal cavity. The atom and field become dynamically entangled by their interaction with the state of the combined system after the interaction time t

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} a_n(t)\,|2,n\rangle + b_n(t)\,|1,n\rangle \tag{3}$$

to be expanded over the Fock field states $|n\rangle$, $n = 0, 1, \ldots$. Here $a_n(t) \equiv \alpha_n(t) + i\beta_n(t)$ and $b_n(t) \equiv \rho_n(t) + i\eta_n(t)$ are the complex-valued probability amplitudes to find the field in the state $|n\rangle$ and the atom in the states $|2\rangle$ and $|1\rangle$, respectively. In the process of emitting and absorbing photons, atoms not only change their internal electronic states but their external translational states change as well due to the photon recoil effect. If atoms are not too cold and their average momenta are large as compared to the photon momentum $\hbar k_f$, one can describe the translational degree of freedom classically. The whole dynamics is now governed by the Hamilton-Schrödinger equations [12] that have the following normalized form in the frame rotating with the frequency $\omega_f(n + 1/2)$:

$$\begin{aligned}
\dot{x} &= \kappa p, \\
\dot{p} &= -2\sin x \sum_{n=0}^{\infty} \sqrt{n+1}\left(\alpha_n\rho_{n+1} + \beta_n\eta_{n+1}\right), \\
\dot{\alpha}_n &= -\frac{\delta}{2}\beta_n - \sqrt{n+1}\eta_{n+1}\cos x, \\
\dot{\beta}_n &= \frac{\delta}{2}\alpha_n + \sqrt{n+1}\rho_{n+1}\cos x, \\
\dot{\rho}_{n+1} &= \frac{\delta}{2}\eta_{n+1} - \sqrt{n+1}\beta_n\cos x, \\
\dot{\eta}_{n+1} &= -\frac{\delta}{2}\rho_{n+1} + \sqrt{n+1}\alpha_n\cos x,
\end{aligned} \tag{4}$$

where $x = k_f\langle\hat{x}\rangle$ and $p = \langle\hat{p}\rangle/\hbar k_f$ are the atomic center-of-mass position and momentum, respectively. Dot denotes differentiation with respect to dimensionless time

$\tau = \Omega_0 t$, where Ω_0 is the amplitude coupling constant. The normalized recoil frequency, $\kappa = \hbar k_f^2/m_a\Omega_0 \ll 1$, and the atom-field detuning, $\delta = (\omega_f - \omega_a)/\Omega_0$, are the control parameters.

Inspite of existence of an infinite number of the integrals of motion

$$R_n = \alpha_n^2 + \beta_n^2 + \rho_{n+1}^2 + \eta_{n+1}^2 = \text{const}, \quad \sum_{n=0}^{\infty} R_n \leqslant 1 \quad (5)$$

and conservation of the total energy

$$W = \frac{\kappa p^2}{2} - \frac{\delta}{2}\sum_{n=0}^{\infty}\left(\alpha_n^2 + \beta_n^2 - \rho_{n+1}^2 - \eta_{n+1}^2\right) - \\ - 2\cos x \sum_{n=0}^{\infty}\sqrt{n+1}\left(\alpha_n\rho_{n+1} + \beta_n\eta_{n+1}\right), \quad (6)$$

the Hamilton-Schrödinger system (4) is, in general, nonintegrable. The type of the center-of-mass motion depends strongly on the values of the detuning δ. In the limit of zero detuning and with initially excited or deexcited atoms, the optical potential disappears, and atoms move with a constant velocity $\dot{x} = \kappa p_0$. The quantum evolution is periodic with the period $\pi/\kappa p_0$, and exact solutions for purity, von Neumann entropy, fidelity $f(\tau)$, and other quantum characteristics can be found in the explicit forms. For example, the atomic population inversion at $\delta = 0$ is the following:

$$z(\tau) = \sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} z_n(0)\cos\left(\frac{2\sqrt{n+1}}{\kappa p_0}\sin\kappa p_0\tau\right), \\ z_n = \alpha_n^2 + \beta_n^2 - \rho_{n+1}^2 - \eta_{n+1}^2. \quad (7)$$

With the detuning being large, $|\delta| \gg 0$, the optical potential is shallow, atom moves with almost a constant velocity, $\simeq \kappa p_0$, slightly modulated by the standing wave, and its inversion oscillates with a small depth (excepting for the case of the so-called Doppler-Rabi resonance with maximal Rabi oscillations that occur at the condition $|\delta| = \kappa p_0$ [13]). If the atomic kinetic energy, $\kappa p^2/2$, is not enough to overcome barriers of the optical potential, the atomic center of mass oscillates in one of the potential wells.

Numerical simulation shows that there exist conditions, defined mainly by the values of the detuning, when atoms move chaotically in a cavity. This type of motion may be called a chaotic or random walking, and it is quantified by positive values of the maximal Lyapunov exponent λ. In Fig. 1 we show by the dotted line the dependence $\lambda(\delta)$ computed with Eqs. (4) and the following initial conditions: $x_0 = 0$, $p_0 = 25$, the atom is prepared in the state $|2\rangle$ and the field is initially in a coherent state with the average number of photons $\bar{n} = 10$. The normalized recoil frequency is chosen to be $\kappa = 0.001$, a reasonable value with usual atoms in a high-quality optical microcavity in the strong-coupling limit. Stability

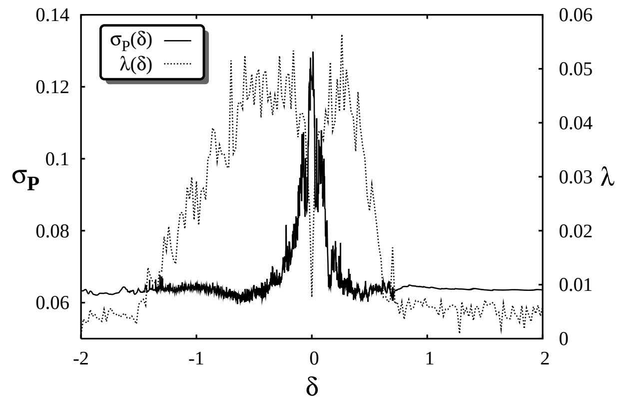

FIG. 1: Quantum-classical correlation between the dependencies of the variance of quantum purity, σ_P and the maximal Lyapunov exponent λ (in units of Ω_0) on the atom-field detuning δ (in units of Ω_0).

of the computation with respect to truncating the set (4) was checked. In most the cases $n = 100$ was taken.

The entanglement between the internal atomic and field degrees of freedom can be characterized by the quantity known as purity

$$P(\tau) = \mathrm{Tr}_a\, \rho_a^2(\tau), \quad (8)$$

where $\rho_a(\tau)$ is the reduced atomic density matrix

$$\rho_a(\tau) = \sum_{n=0}^{\infty} \langle n|\rho(\tau)\,|n\rangle \quad (9)$$

with the total density matrix to be $\rho(\tau) = |\Psi(\tau)\rangle\,\langle\Psi(\tau)|$. Purity is maximal if an atom is in one of its energetic states $|1\rangle$ or $|2\rangle$; i. e. $P_{\max} = \mathrm{Tr}_a\,\rho_a^2 = \mathrm{Tr}_a\,\rho_a = 1$. Purity is minimal if $\rho_a = I/2$, i. e. $P_{\min} = 1/2$, where I is the identity matrix. In terms of the probability amplitudes, it is given by

$$P = \left(\sum_{n=0}^{\infty}\left(\alpha_n^2 + \beta_n^2\right)\right)^2 + \left(\sum_{n=0}^{\infty}\left(\rho_n^2 + \eta_n^2\right)\right)^2 + \\ +2\left(\sum_{n=0}^{\infty}\left(\alpha_n\rho_n + \beta_n\eta_n\right)\right)^2 - 2\left(\sum_{n=0}^{\infty}\left(\alpha_n\eta_n + \beta_n\rho_n\right)\right)^2. \quad (10)$$

The root mean square variance of purity, $\sigma_P = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$ has been computed in the range of the detuning $|\delta| \leqslant 2$ at the same conditions as it was done in computing the maximal Lyapunov exponent λ. Irregular oscillations of σ_P occurs on the same interval of $|\delta| \lesssim 1$, where $\lambda > 0$ (see Fig. 1). Computing the von Neumann entropy, $S = -\mathrm{Tr}_a(\rho_a \ln \rho_a)$, we have found the same correlations of its variance with λ.

The chaotic centre-of-mass walking has fractal properties. Placing atoms at the point $x = 0$ with the same

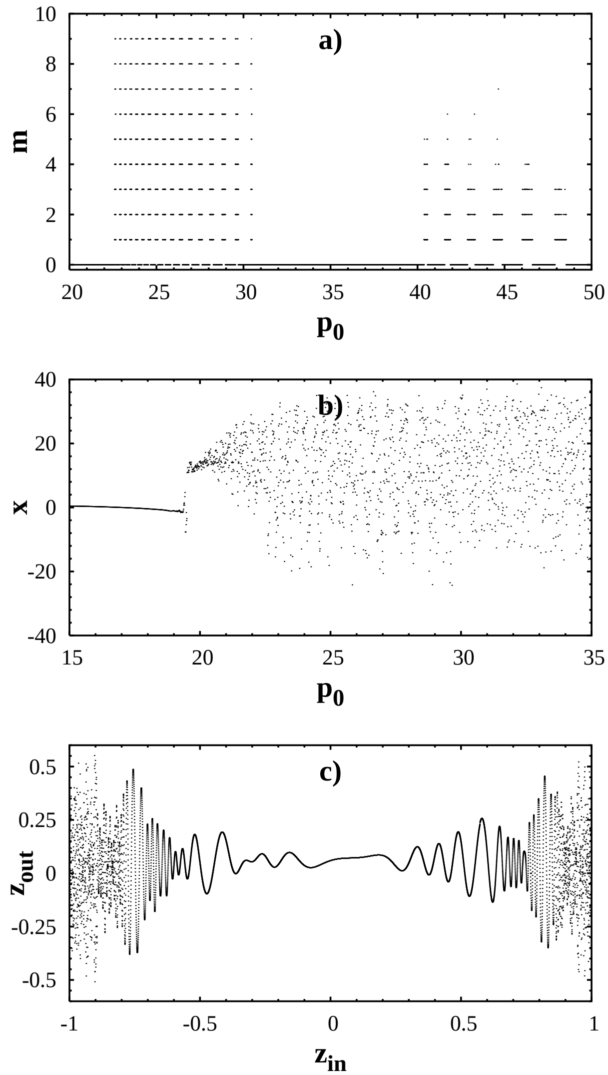

FIG. 2: (a) Fractal set of the initial momenta p_0 (in units of $\hbar k_f$) of atoms that leave a one-wave length cavity after m turns. (b) Sensitive dependence of the atomic position x (in units k_f^{-1}) on the initial momentum p_0. (c) Sensitive dependence of the output values of the atomic population inversion $z_{\rm out}$ on its initial values $z_{\rm in}$. Control parameters $\delta = 0.4$ and $\kappa = 0.001$.

initial conditions and parameters but with different values of initial momenta p_0 , we compute the time $T(p_0)$, the atom with a given value p_0 needs to reach one of the nodes of the standing wave at $x = -\pi/2$ and $x = 3\pi/2$, and the number of times, m, when it changes its direction of motion. The scattering function $T(p_0)$ is found to have a self-similar structure with singularities on a Cantor-like set of initial values of momenta p_0. In Fig. 2a we demonstrate the mechanism of generating this set at $\delta = 0.4$. There are two sets of atomic trajectories with $T \to \infty$, the countable one consisting of separatrix-like trajectories, corresponding to the ends of the intervals in Fig. 2a, and the uncountable one consisting of trajectories with $m = \infty$. The chaotic motion can be, in principle, verified in experiments on 1 D-scattering of atoms at the standing wave. Fig. 2b demonstrates sensitive dependence of the atomic positions on p_0 at a fixed time moment. A smooth segment of this function in the range $p_0 \lesssim 20$ should be attributed to atomic oscillations in the first well of the optical potential since these values of p_0 are not enough to overcome the respective potential barrier. When p_0 exceeds a critical values, atoms leave the well, and it is practically impossible to predict even the sign of the atomic position. The so-called predictability horizon can be estimated as follows: $\tau_p \simeq \lambda^{-1} \ln(\Delta x/\Delta x_0)$, where Δx is the confidence interval and Δx_0 the initial inaccuracy in preparing initial atomic positions. In order to demonstrate the quantum-classical correspondence in the chaotic regime we compute the dependence of the output values of the atomic population inversion $z_{\rm out}$ at a fixed moment on its initial values $z_{\rm in}$ with the other conditions and parameters to be the same. Fig. 2c illustrates that this function is regular in the range $|z_{\rm in}| \lesssim 0.5$, where the atomic center-of-mass motion is regular, and is irregular if $0.5 \lesssim |z_{\rm in}| \leqslant 1$ where atoms move chaotically.

To quantify instability of quantum evolution in cavity QED we compute the decay of the fidelity $f(\tau)$ which is the overlap (1) of two states $|\Psi_1(\tau)\rangle$ and $|\Psi_2(\tau)\rangle$, identical at $\tau = 0$, that evolve under two Hamiltonians (2) with the slightly different detunings $\Delta\delta = \delta_1 - \delta$.

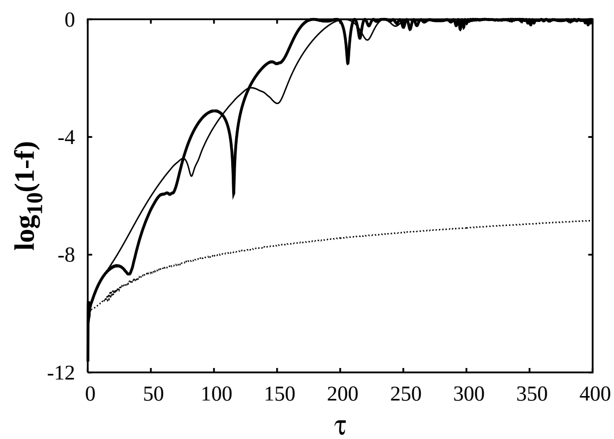

FIG. 3: Decay of the fidelity of quantum motion $1 - f(\tau)$ (logarithmic scale) in the chaotic (thick and thin lines) and regular (dotted line) regimes. Time τ is in units of Ω_0^{-1}.

We have found previously (see Fig. 2) that with the initial momentum $p_0 = 25$ and $\delta = 0.4$ the type of atomic motion depends strongly on the initial atomic inversion $z(0)$. If an atom is prepared initially in one of its energetic states, i. e. $z(0) = \pm 1$, its classical and quantum dynamics are unstable, whereas they are stable with $z(0) = 0$ under the same other conditions. In Fig. 3 we show for convenience the decay of the quantity $\log_{10}(1 - f)$ in the regimes of chaotic walking (thick and thin lines, $z(0) = \pm 1$) and regular motion (dotted line, $z(0) = 0$) with $\Delta\delta = 10^{-4}$. In the chaotic regime the fidelity decays exponentially with the rate $\lambda \simeq 0.04$ to

be equal to the maximal Lyapunov exponent computed with the set (4). This result does not depend on the values of differences in the control parameters $\Delta\delta$. The fidelity practically does not decay in the regular regime at $z(0) = 0$, and the respective maximal Lyapunov exponent was computed to be zero.

We emphasize that sensitive dependence of quantum motion both on initial states and parameters may arise with an atom in a quantized cavity field. Single chaotic degree of freedom, arising naturally when we take into account photon recoils, provides quantum instability and irreversibility. We do not need an infinite bath or any kind of noise for that. The quantum instability has been shown to be quantified by the respective maximal Lyapunov exponent providing a quantum-classical correspondence.

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[1] B.V. Chirikov, Chaos **1**, 95 (1991).
[2] W.H. Zurek, Phys. Today **44**, 36 (1991).
[3] D. Giulini et al., *Decoherence and the Appearance of a Classical World in Quantum Theory*, 2 nd ed. (Springer, New York, 2003).
[4] R. Blume-Kohout and W.H. Zurek, Phys. Rev. A **68**, 032104 (2003).
[5] A. Peres, Phys. Rev. A **30**, 1610 (1984).
[6] R. Schack and C.M. Caves, Phys. Rev. E **53**, 3257 (1996).
[7] R.A. Jalabert and H.M. Pastawski, Phys. Rew. Lett. **86**, 2490 (2001).
[8] Ph. Jacquod, P.G. Silvestrov, and C.W.J. Beenakker, Phys. Rev. E **64**, 055203 (2001).
[9] G. Benenti and G. Casati, Phys. Rev. E **65**, 066205 (2002).
[10] T. Prosen, T.H. Seligman, and M. Znidaric, Phys. Rev. A **67**, 042112 (2003).
[11] E.T. Jaynes and E.W. Cummings, Proc. IEEE **51**, 89 (1963).
[12] S.V. Prants and M.Yu. Uleysky, Phys. Lett. A **309**, 357 (2003).
[13] M.Yu. Uleysky, L.E. Kon'kov, and S.V. Prants, Comm. Nonlin. Sci. Numer. Simul. **8**, 329 (2003).